A benchmark of a viscoelastic(LUBBY2) model

The LUBBY2 model is based on the generalised Burgers model and is described by the following evolution equation [1]:

$$\boldsymbol{\sigma} = K_\mathrm{M} e \mathbf{I} + 2G_\mathrm{M}\left[\boldsymbol{\epsilon}^\mathrm{D} - \boldsymbol{\epsilon}^\mathrm{D}_\mathrm{M} - \boldsymbol{\epsilon}^\mathrm{D}_\mathrm{K}\right]$$

$$\dot{\boldsymbol{\epsilon}}^\mathrm{D}_\mathrm{K} = \frac{1}{2\eta_\mathrm{K}}\left(\boldsymbol{\sigma}^\mathrm{D} - 2G_\mathrm{K}\boldsymbol{\epsilon}^\mathrm{D}_\mathrm{K}\right) \tag{1}$$

$$\dot{\boldsymbol{\epsilon}}^\mathrm{D}_\mathrm{M} = \frac{1}{2\eta_\mathrm{M}}\boldsymbol{\sigma}^\mathrm{D}$$

where $\boldsymbol{\sigma}^\mathrm{D}$ is the deviatoric stress, $\boldsymbol{\epsilon}^\mathrm{D}$ is the deviatoric strain, and e is the volume strain. The viscosities and the Kelvin shear modulus of the Lubby2 formulation are functions of the current stress state

$$\eta_\mathrm{M} = \eta_\mathrm{M0} e^{m_1 \sigma_\mathrm{eff}}$$

$$\eta_\mathrm{K} = \eta_\mathrm{K0} e^{m_2 \sigma_\mathrm{eff}} \tag{2}$$

$$G_\mathrm{K} = G_\mathrm{K0} e^{m_\mathrm{G} \sigma_\mathrm{eff}}$$

with

$$\sigma_{eff} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}_\mathrm{D} : \boldsymbol{\sigma}_\mathrm{D}} \tag{3}$$

where m_a are material parameters characterising the stress dependency.
The rheological model is shown in Fig. 1 consisting of a Maxwell element in series with a Kelvin element.

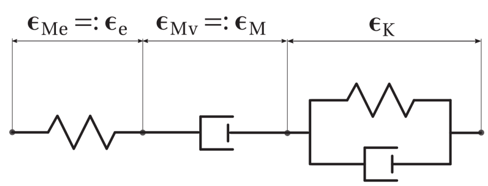


Figure 1: Rheological analogue of the LUBBY2 model.

The state vector $z = \left(\boldsymbol{\sigma}^\mathrm{DT}, \boldsymbol{\epsilon}^\mathrm{DT}_\mathrm{K}, \boldsymbol{\epsilon}^\mathrm{DT}_\mathrm{M}\right)$, which contains unknowns, are solved by using the Newton-Raphson method with the residual vector of

$$\mathbf{r}^j_1 = \boldsymbol{\sigma}^{\mathrm{D}j} - 2\left(\boldsymbol{\epsilon}^{\mathrm{D}j} - \boldsymbol{\epsilon}^{\mathrm{D}j}_K - \boldsymbol{\epsilon}^{\mathrm{D}j}_M\right) \tag{4}$$

$$\mathbf{r}^j_2 = \frac{\boldsymbol{\epsilon}^{\mathrm{D}j}_\mathrm{K} - \boldsymbol{\epsilon}^{\mathrm{D}t}_\mathrm{K}}{\Delta t} - \frac{1}{2\eta_\mathrm{K}}\left(G_\mathrm{M}\boldsymbol{\sigma}^{\mathrm{D}j} - 2G_\mathrm{K}\boldsymbol{\epsilon}^{\mathrm{D}j}_\mathrm{K}\right) \tag{5}$$

$$\mathbf{r}^j_3 = \frac{\boldsymbol{\epsilon}^{\mathrm{D}j}_M - \boldsymbol{\epsilon}^{\mathrm{D}t}_M}{\Delta t} - \frac{G_\mathrm{M}}{2\eta_\mathrm{M}}\boldsymbol{\sigma}^{\mathrm{D}j} \tag{6}$$

and the 18 × 18 Jacobian:

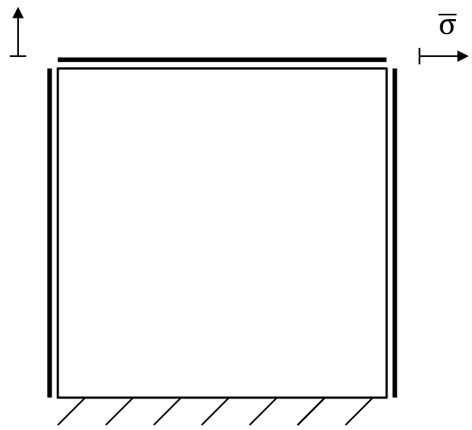


Figure 2: Loading and boundary conditions.

Table 1: Material properties used in the LUBBY2 model

G_{M0} / MPa	K_{M0} / MPa	η_{M0} / (MPa s)	G_{K0} / MPa	η_{K0} / (MPa s)	m_1	m_2	m_{G}
0.8	0.8	0.5	0.8	0.5	−0.3	−0.2	−0.2

$$\frac{\partial \mathbf{r}}{\partial \mathbf{z}} = \begin{pmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} & \boldsymbol{J}_{13} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} & \boldsymbol{J}_{23} \\ \boldsymbol{J}_{31} & \boldsymbol{J}_{32} & \boldsymbol{J}_{33} \end{pmatrix} \tag{7}$$

where the components are given as follows:

$$\boldsymbol{J}_{11} = \frac{1}{\Delta t}\boldsymbol{I},\ \boldsymbol{J}_{12} = \frac{2}{\Delta t}\boldsymbol{I},\ \boldsymbol{J}_{13} = \frac{2}{\Delta t}\boldsymbol{I} \tag{8}$$

$$\boldsymbol{J}_{21} = -\frac{G_{\mathrm{M}}}{2\eta_K}\boldsymbol{I},\ \boldsymbol{J}_{22} = \frac{1}{\Delta t}\boldsymbol{I} + \frac{G_K}{\eta_K}\boldsymbol{I},\ \boldsymbol{J}_{23} = \mathbf{0} \tag{9}$$

$$\boldsymbol{J}_{31} = -\frac{G_{\mathrm{M}}}{2\eta_{\mathrm{M}}}\boldsymbol{I},\ \boldsymbol{J}_{32} = \mathbf{0},\ \boldsymbol{J}_{33} = \frac{1}{\Delta t}\boldsymbol{I} \tag{10}$$

for $\sigma_{\mathrm{eff}} > 0$

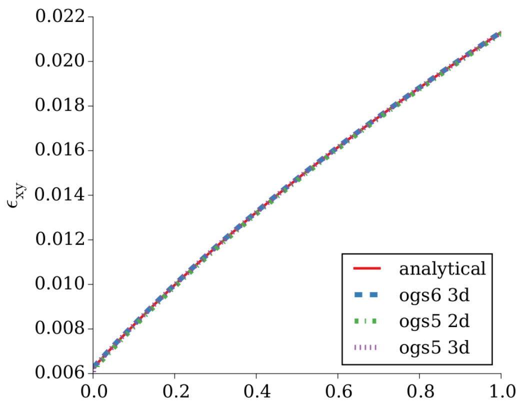
$$\boldsymbol{J}_{21} = \frac{1}{2\eta_K}\left(G_{\mathrm{M}}\boldsymbol{\sigma}^{\mathrm{D}j} - 2G_{\mathrm{K}}\boldsymbol{\epsilon}_{\mathrm{K}}^{j}\right)\frac{3}{2}m_2 G_{\mathrm{M}}\frac{\left(\boldsymbol{\sigma}^{\mathrm{D}j}\right)^T}{\sigma_{\mathrm{eff}}} + \frac{3}{2\eta_K}\boldsymbol{\epsilon}_{\mathrm{K}}^{j} m_{\mathrm{G}} G_{\mathrm{K}} G_{\mathrm{M}}\frac{\left(\boldsymbol{\sigma}^{\mathrm{D}j}\right)^T}{\sigma_{\mathrm{eff}}} \tag{11}$$

$$\boldsymbol{J}_{31} = \frac{1}{2\eta_M}G_{\mathrm{M}}\boldsymbol{\sigma}^{\mathrm{D}j}\frac{3}{2}m_1 G_{\mathrm{M}}\frac{\left(\boldsymbol{\sigma}^{\mathrm{D}j}\right)^T}{\sigma_{\mathrm{eff}}} \tag{12}$$

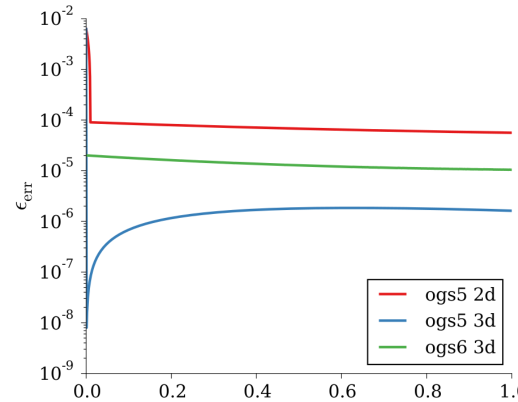
The mechanical model is a square plate/cube with a positive shear stress of 0.01 MPa applied on the top side/surface, see Fig. 2. Displacements of the left, right side and the top are constrained in vertical direction. The material property set for this benchmark is listed in Table 1.

References

[1] Thomas Nagel, Uwe Jens Görke, Kevin M. Moerman, and Olaf Kolditz. On advantages of the Kelvin mapping in finite element implementations of deformation processes. *Environmental Earth Sciences*, 75(11), 2016.



(a) ϵ_{xy}



(b) deviation

Figure 3: Variation of the shear strain with time (a) and the deviation between analytical solution and numerical simulations (b).